PG – 254

II Semester M.Sc. Degree Examination, June/July 2014
(RNS) (2011 – 12 & Onwards)
MATHEMATICS
M – 203 : Functional Analysis

Time : 3 Hours   Max. Marks : 80

Instructions : *1) Answer* ***any five*** *questions choosing atleast* ***two*** *from* ***each*** *Part.*
2) ***All*** *questions carry* ***equal*** *marks.*

PART – A

1. a) Define a Banach Space. Show that a non-zero normed linear space N is a Banach space if and only if the set $\{x \mid \|x\| = 1\}$ is complete. **8**

   b) Let B be a Banach space and let M and N be linear subspaces of B. If $B = M \oplus N$ and $\|z\|_1 = \|x\| + \|y\|$ then prove that $\|\bullet\|_1$ is a norm on B. Further prove that $(B, \|\bullet\|_1)$ is complete if M and N are closed in B. **8**

2. a) Show that the set B (N, N′) of all continuous linear transformations a normed linear space from N to N′ is a complete normed linear space if N′ is complete. **10**

   b) If S is a continuous linear transformation of a normed linear space N to a normed linear space N′ and if M is its null space then show that S induces a natural linear transformation S′ of $\frac{N}{M}$ into N′ such that $\|S\| = \|S'\|$. **6**

3. a) Let M be a linear subspace of a normed linear space N. If $x_0 \notin M$, $x_0 \in N$ and if $M_0 = M \in [x_0]$, then prove that f can be extended to a functional f_0 on M_0 such that $\|f_0\| = \|f\|$. **9**

   b) Let M be a closed linear subspace of a normed linear space N and let $x_0 \notin M$. If d is the distance from x_0 to M, show that there exists $h \in N^*$ such that that h (x_0) = 1, $\|h\| = Y_d$. **7**

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4. a) Show that there is a natural embedding of N into N** obtained by the isometric isomorphism $x \to F_x$. $\left(\text{where } F_x :: N^* \to F : F_x(f) = f(x), \forall\, x \in N, f \in N^*\right)$. **8**

b) State and prove closed graph theorem. **8**

PART – B

5. a) Define a Hilbert Space. Show that the following inequalities hold in a Hilbert space H.

i) $|< x, y >| \leq \|x\| \|y\|$

ii) $\|x + y\| \leq \|x\| + \|y\|$, for all x, $y \in H$. **6**

b) Let M be a closed linear subspace of a Hilbert space H. If $x \in H$ and $x \notin M$ then prove that there is a unique vector $y_0 \in M$ such that $\|x - y_0\| = d(x, M)$. **5**

c) If M and N are closed linear subspaces of a Hilbert space H such that $M \perp N$ then prove that $M + N$ is a closed linear subspace. **5**

6. a) Prove that every nonzero Hilbert space contains a complete orthonormal set. **5**

b) Let H be a Hilbert space and $\{e_i\}$ be an orthonormal set in H. Then prove that the following are equivalent :

i) $\{e_i\}_{i=1}^{n}$ is complete

ii) $x \perp e_i, \forall i \Rightarrow x = 0$

iii) $x \in H \Rightarrow x = \sum_{i=1}^{n} \langle x, e_i \rangle e_i$

iv) $x \in H \Rightarrow \|x\|^2 = \sum_{i=1}^{n} |\langle x, e_i \rangle|^2$. **8**

c) Show that in a Hilbert space H, each $y \in H$ gives rise to a $f_y \in H^*$. **3**

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7. a) Show that the self adjoint operators is B (H), where H is a Hilbert space form a closed real linear space. **5**

b) Show that an operator T on a Hilbert space H is self adjoint if and only if $\langle Tx, x \rangle$ is real for all $x \in H$. **6**

c) Show that an operator $T \in B(H)$ is normal if and only if its real and imaginary parts commute. **5**

8. a) If $P_1, P_2, \ldots, P_n$ are projections on closed linear subspaces $M_1, M_2, \ldots, M_n$ of a Hilbert space H then prove that $P_1 + P_2 + \ldots + P_n$ is a projection if and only if $\{P_i\}$ are pair wise orthogonal. **8**

b) State and prove spectral theorem. **8**

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